**Module 5: Conic Sections and Polar Coordinates**

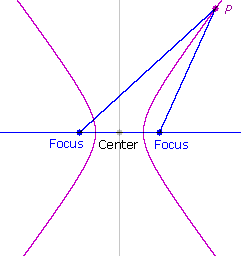
**III. The Hyperbola**

After completing this section, you should be able to:

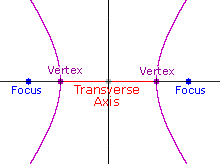
* identify the center, vertices, foci, transverse axis, conjugate axis, and asymptotes of a hyperbola
* transform an equation of a hyperbola into standard form
* graph hyperbolas

**Geometric Definition of a Hyperbola**

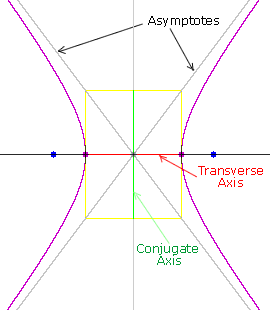
A **hyperbola** is the set of all points in the plane for which the absolute value of the difference of the distances from two fixed points (**foci**) is constant.



The **center** of a hyperbola is the midpoint of the line segment between the foci.



A hyperbola has two perpendicular axes of symmetry, which intersect at the center of the hyperbola. One line of symmetry passes through the foci and intersects with the hyperbola in two points called the **vertices**. The **transverse axis** is the line segment between the vertices.



The **conjugate axis** of a hyperbola is a line segment that is perpendicular to the transverse axis and contains the center. (The endpoints of the conjugate axis are defined in the tables below.)

Together, the transverse axis and the conjugate axis determine a rectangle whose center is the center of the hyperbola. The diagonals of the rectangle determine two lines through the center of the hyperbola. These two lines are the **asymptotes** of the hyperbola. As you move along either one of the two hyperbolic paths, farther and farther away from a vertex, you get closer and closer to an asymptote.

The standard equations and properties of hyperbolas with center at the origin are summarized below.

|  |  |
| --- | --- |
| **Standard Equation of a Hyperbola with Center (0, 0)** | |
| **Horizontal Transverse Axis** | |
| Standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Transverse-stnd-formula.gif  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/cntr-of-hyperla1.png   |  |  |  | | --- | --- | --- | | Axis | Orientation | Selected Points on Axis | | Transverse | Horizontal | Vertices (–*a*, 0) and (*a*, 0)  Foci (–*c*, 0) and (*c*, 0) where *c*2 = *a*2 + *b*2 | | Conjugate | Vertical | Endpoints (0, –*b*) and (0, *b*) |   Asymptotes: https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/asymtotes-b-ovr-a.gif and https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/asymtotes-b-ovr-a2.gif |  |
| **Vertical Transverse Axis** |  |
| Standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/VTA-stndversion.gif  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/cntr-of-hyperla2.png   |  |  |  | | --- | --- | --- | | Axis | Orientation | Selected Points on Axis | | Transverse | Vertical | Vertices (0, –*a*) and (0*, a*) Foci (0, –*c*) and (0*, c*) where *c*2 = *a*2 + *b*2 | | Conjugate | Horizontal | Endpoints (–*b,*0) and (*b,*0) |   Asymptotes: https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/asymtotes-a-ovr-b.gifand https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/asymtotes-a-ovr-b2.gif |  |

**Example III.1:** Find the center, vertices, foci, and asymptotes of the hyperbola given by the equation

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and graph the hyperbola.

**Solution:**The center of the hyperbola is the point (0, 0). *a* = 2 and *b* = 1.

Find *c*:

*c*2 = *a*2 + *b*2  
*c*2 = (2)2 + (1)2  
*c*2 = 5  
*c* =square root of 5

Since the coefficient of *y*2 is positive and the coefficient of the *x*2 term is negative, the transverse axis is vertical.

The vertices are located 2 units above and below the center:

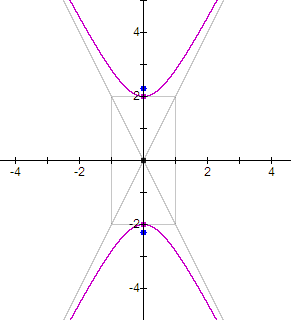
*V*1: (0, –2) and *V*2: (0, 2).

The foci are located square root of 5 units above and below the center:

*F*1: (0, –square root of 5) and *F*2: (0,square root of 5).

The endpoints of the conjugate axis are located 1 unit left and right of the center:

*E*1: (–1, 0) and *E*2: (1, 0).



To sketch the graph, first plot the vertices (0, –2) and (0, 2) and the endpoints (–1, 0) and (1, 0) of the conjugate axis.

Draw the central rectangle determined by these points. Draw the diagonals of the rectangle. The extensions of the diagonals are the asymptotes.

Since the height of the rectangle is 2*b* = 4 and the width of the rectangle is 2*a* = 2, the slopes of the asymptotes are ±4/2 = 2 and –2. The lines pass through the center (0, 0), so the equations of the asymptotes, in point-slope form, are *y* = 2*x* and *y* = –2*x*.

Now suppose the center of a hyperbola is (*h*, *k*) rather than (0, 0). Start with a standard equation of a hyperbola with center (0, 0) and replace *x* with *x* – *h* and *y* with *y* – *k* to arrive at the standard form of a hyperbola with center (*h*, *k*). The results are summarized below.

|  |
| --- |
| **Standard Equation of a Hyperbola with Center *C*: (*h*, *k*)** |
| **Transverse Axis Parallel to the *x*-Axis** |
| Standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/standard-formla-4-hyperbola.gif  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/cntr-of-hyperla-F1F2.png  Center *C*: (*h*, *k*)   |  |  |  | | --- | --- | --- | | Axis | Orientation | Selected Points on Axis | | Transverse | Horizontal | Vertices *V*1: (*h* – *a*, *k*) and *V*2: (*h* + *a*, *k*) Foci *F*1: (*h* – *c*, *k*) and *F*2: (*h* + *c*, *k*) where *c*2 = *a*2 + *b*2 | | Conjugate | Vertical | Endpoints *E*1: (*h*, *k* – *b*) and  *E*2: (*h*, *k* + *b*) |   Asymptotes: https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/asymptote-y-k-b-ovr%20a.gif and https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/asymptote-y-k--b-ovr%20a.gif |
| **Transverse Axis Parallel to the *y*-Axis** |
| Standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/tranvse-vertical-2yaxis.gif  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/cntr-of-hyperla-E1E2.png  Center *C*: (*h*, *k*)   |  |  |  | | --- | --- | --- | | Axis | Orientation | Selected Points on Axis | | Transverse | Vertical | Vertices *V*1: (*h*, *k*– *a*) and *V*2: (*h*, *k* + a)  Foci *F*1: (*h*, *k* – *c*) and *F*2: (*h*, *k* + *c*) where *c*2 = *a*2 + *b*2 | | Conjugate | Horizontal | Endpoints *E*1: (*h*– *b*, *k*) and *E*2:(*h* + *b*, *k*) |   Asymptotes: https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/asymtotes-a-ovr-b3.gif and https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/asymtotes-a-ovr-b4.gif |

**Example III.2:** Find the center, vertices, foci, and asymptotes of the hyperbola given by the equation 4*x*2 – 9*y*2 – 32*x* + 18*y* + 19 = 0 and graph the hyperbola.

Solution:

|  |  |
| --- | --- |
| 4*x*2 – 9*y*2 – 32*x* + 18*y* + 19 = 0 |  |
| (4*x*2 – 32*x*) – (9*y*2 – 18*y*) + 19 = 0 | Group terms. |
| 4(*x*2 – 8*x*) – 9(*y*2 – 2*y*) + 19 = 0 | Factor. |
| 4(*x*2 – 8*x* + 16– 16) – 9(*y*2 – 2*y* + 1 – 1) + 19 = 0 | Complete the square. |
| 4(*x*2 – 8*x* + 16)  – 64 – 9(*y*2 – 2*y*  + 1) + 9 + 19 = 0 | Multiply and regroup. |
| 4(*x* – 4)2 – 9(*y* – 1)2 – 36 = 0 | Factor and simplify. |
| 4(*x* – 4)2 – 9(*y* – 1)2 = 36 | Rewrite in standard form. |
| https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod-sec-III-exIII-2a.gif | Divide by 36. |
| https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod-sec-III-exIII-2b.gif | Simplify. |
| https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/mod5-II-b3d.gif = 1 | Rewrite in standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/mod5-I-d-2a-red.gif. |

The center of the hyperbola is the point (4, 1). The transverse axis is horizontal. *a* = 3 and *b* = 2.

The vertices are located 3 units to the left and the right of the center (4, 1):

*V*1: (4 – 3, 1) = (1, 1) and *V*2: (4 + 3, 1) = (7, 1)

The endpoints of the conjugate axis are located 2 units below and above the center (4, 1):

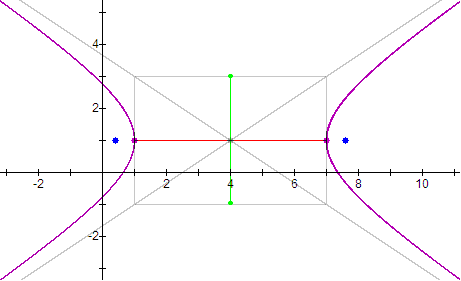
*E*1: (4, 1 – 2) = (4, –1) and *E*2: (4, 1 + 2) = (4, 3)

Find *c*:

*c*2 = *a*2 + *b*2  
*c*2 = (3)2 + (2)2  
*c*2 = 9 + 4  
*c*2 = 13  
 *c* =https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/sqrt13.gif

The foci are located https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/sqrt13.gif units to the left and the right of the center (4, 1):

*F*1: (4, –https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/sqrt13.gif, 1) and *F*2: (4, +https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/sqrt13.gif, 1) .



To sketch the graph, first plot the vertices (1, 1) and (7, 1) and the endpoints (4, –1) and (4, 3) of the conjugate axis.

Draw the central rectangle determined by these points. Draw the diagonals of the rectangle. The extensions of the diagonals are the asymptotes.

Because the height of the rectangle is 2*b* = 4 and the width of the rectangle is 2*a* = 6, the slopes of the asymptotes are ±4/6 = 2/3 and –2/3. The lines pass through the center (4, 1), so the equations of the asymptotes, in point-slope form, are *y* – 1 = (2/3)(*x* – 4) and *y* – 1 = –(2/3)(*x* – 4).

Although planets and some comets travel in elliptical orbits, other comets travel hyperbolic paths, following one of the two portions of a hyperbola, with the sun at one focus. Whereas comets on elliptical paths revolve about the sun, comets on hyperbolic paths travel by the sun just one time.

Before concluding the discussion of conic sections, take a moment to recall the general second-degree equation for a conic section, *Ax*2 + *Bxy* + *Cy*2 + *Dx* + *Ey* + *F* = 0.

All of the second-degree equations you have examined in this module are lacking an *xy* term (that is, *B* = 0), and all of the corresponding graphs have axes of symmetry that are parallel to the *x* or *y* axes. The presence of an *xy* term indicates that the axes of symmetry are not parallel to the *x* or *y* axes; they have been rotated. The study of equations involving rotations is beyond the scope of this course.

Now, consider a second-degree equation without an *xy* term: *Ax*2 + *Cy*2 + *Dx* + *Ey* + *F* = 0.

If you are given an equation of this form that you know represents a conic section, you can identify the type of conic section by examining the coefficients *A* and *C*.

* If *A* or *C* (but not both) is 0, then there is only one squared variable, and the conic section is a parabola.
* If *A* = *C*, the section conic is a circle.
* If *A* and *C* have the same sign (*AC* > 0) but *A* ≠ *C*, the conic section is an ellipse.
* If *A* and *C* have opposite signs (*AC* < 0), the conic section is a hyperbola.

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